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On the stability loss of a bimaterial finite sized body with a tunnel crack in an interface

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Abstract

The stability loss of a finite sized layered system subjected to a uniform compressive load parallel to the free surface and containing an interface crack is considered. The eigenvalue problem for the system of homogeneous Cauchy-type singular integral equations of the second kind is formulated and the critical value of the load is found numerically by utilizing Gauss–Chebyshev integral formula. Numerical results for the critical load are presented for various geometrical parameters and material properties of both layers.

Keywords: Stability loss; Bimaterial body; Finite thickness; Interface crack

1. Introduction

Composite structures in some cases can work under the action of a compressive load. Interface cracks in such structures appearing because of manufacturing processes can be the reason of significant reduction of the load-carrying capacity under compressive loads. The presence of interface cracks in such cases may lead to fracture of structures by means of crack opening. The buckling state may occur at relatively small compressive loads due to some sufficiently large embedded delamination. Thus a better understanding of the behavior of the layered systems with interface cracks under a compressive load is quite important for the design of composite structures.

Many investigations have been performed to clarify the influence of a crack or an array of equally spaced co-planar cracks on the compressive strength of infinitesimal homogeneous or composite structures. Keer et al. (1982) utilized exact equilibrium equations for the Jaumann rate of the Kirchhoff stress to estimate the buckling loads of a half-space (or a layer) containing an array of equally spaced co-planar cracks, respectively. The problems of stability loss of infinite homogeneous structures with cracks were considered by Gus (1989) using the methods of complex variable functions and Riemann–Gilbert problems. Wang et al. (1991) considered local buckling problems for a half-space containing a through-the-width crack by the use

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of the stability equation derived from the mathematical theory of elasticity. Recently, Wang and Takao (1995) considered local buckling of a layer bonded to a half-space with an interface crack by means of the method of singular integral equations.

In the present paper the stability loss of a finite sized layered system subjected to a uniform compressive load and containing an interface crack is considered. The effects of the material properties and geometrical parameters of the layers on the local delamination buckling behavior of layered system has been investigated.

2. Formulation of the problem

A plane deformation of two-layered rod $|x| \leqslant L,\ h_1 \leqslant y \leqslant h_2,\ |z| < \infty$ with a tunnel interface crack $|x| \leqslant 2a,\ y = 0$ described in Fig. 1 is considered. The thickness of the upper layer is h_1 and the thickness of the lower one is h_2; $\mu_i,\ v_i\ (i = 1, 2)$ are the shear modulus and Poisson's ratio, respectively. The subscripts, $i = 1, 2$, denote the upper layer and the lower one. The layered system is subjected to a uniform compressive strain ε_0, parallel to the free surfaces $y = h_1$ and $y = h_2$. This strain can be produced by the compression upon the sides $|x| = L$ of two absolutely rigid plates which are in a frictionless contact with the body.

When the compressive load increases and reaches its critical value, an adjacent equilibrium buckled state becomes possible. It means the local delaminated layer deflects into a nonflat configuration and in addition to the initial uniform deformation state, incremental stresses σ_{ij}, strains ε_{ij} and displacements u and v are induced. These increments are assumed to be infinitesimal quantities.

The equilibrium equations for buckling (in the absence of body forces) may be expressed as (Wang and Takao, 1995)

$$
\begin{aligned}
&2(1-v_i)\frac{\partial^2 u_i}{\partial x^2} + (1-2v_i)\frac{\partial^2 u_i}{\partial y^2} + \frac{\partial^2 v_i}{\partial x \partial y} - \frac{P_i(1-2v_i)}{\mu_i}\frac{\partial^2 u_i}{\partial x^2} = 0, \\
&2(1-v_i)\frac{\partial^2 v_i}{\partial y^2} + (1-2v_i)\frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial x \partial y} - \frac{P_i(1-2v_i)}{\mu_i}\frac{\partial^2 v_i}{\partial x^2} = 0,
\end{aligned}
\tag{1}
$$

where

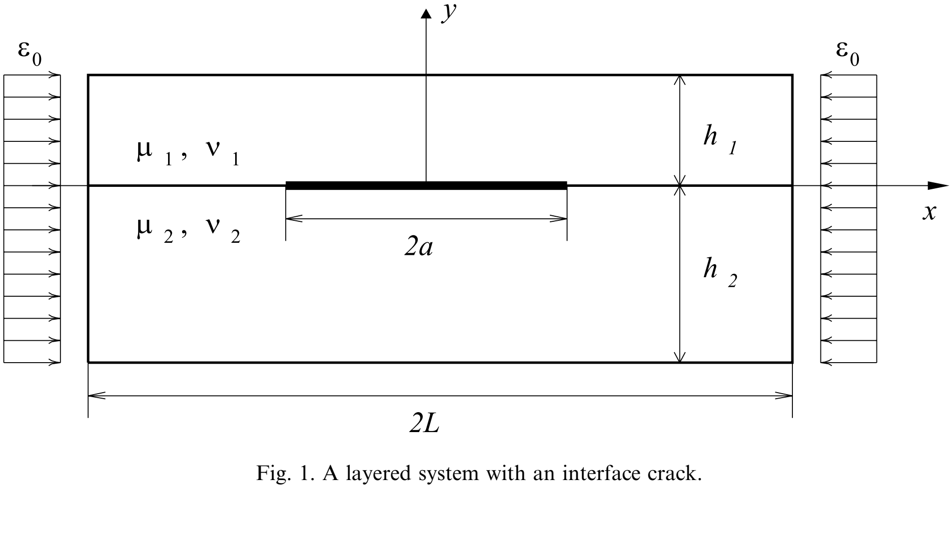

Fig. 1. A layered system with an interface crack.

$$P_i = \frac{2\mu_i}{1 - v_i}\varepsilon_0. \tag{2}$$

Without loss of generality assuming $L = \pi$ the boundary condition for the incremental state can be written as

$$u_i(\pm\pi, y) = 0, \qquad \frac{\partial v_i}{\partial x}(\pm\pi, y) = 0, \quad h_1 \leqslant y \leqslant h_2; \tag{3}$$

$$\begin{aligned}
&\sigma_{1y}(x, h_1) = 0, \qquad \sigma_{1xy}(x, h_1) = 0, \quad |x| < \pi, \\
&\sigma_{2y}(x, -h_2) = 0, \qquad \sigma_{2xy}(x, -h_2) = 0, \quad |x| < \pi, \\
&\sigma_{1y}(x, 0) = \sigma_{2y}(x, 0), \qquad \sigma_{1xy}(x, 0) = \sigma_{2xy}(x, 0), \quad |x| < \pi,
\end{aligned} \tag{4}$$

$$\sigma_{1y}(x, 0) = 0, \qquad \sigma_{1xy}(x, 0) = 0, \quad |x| < a, \tag{5}$$

$$u_1(x, 0) = u_2(x, 0) \qquad v_1(x, 0) = v_2(x, 0), \quad |x| > a. \tag{6}$$

The solution of the system of differential Eq. (1) can be written in the form as Wang and Takao (1995)

$$\begin{aligned}
u_i &= -\frac{\partial^2 \varphi i}{\partial x \partial y}, \\
v_i &= \left(2(1 - v_i) - \frac{P_i(1 - 2v_i)}{\mu_i}\right)\frac{\partial^2 \varphi_i}{\partial x^2} + (1 - 2v_i)\frac{\partial^2 \varphi_i}{\partial y^2} \quad (i = 1, 2),
\end{aligned} \tag{7}$$

where the displacement functions can be presented as

$$\varphi_i(x, y) = \frac{1}{\pi}(A_{i0} + B_{i0} + C_{i0} + D_{i0}) + \frac{2}{\pi}\sum_{p=1}^{\infty}\left(A_{ip}e^{p\alpha_{i1}y} + B_{ip}e^{p\alpha_{i2}y} + C_{ip}e^{-p\alpha_{i1}y} + D_{ip}e^{-p\alpha_{i2}y}\right)\cos px \quad (i = 1, 2). \tag{8}$$

and α_{ij}, $(i, j = 1, 2)$ are the following roots of the associated characteristic equation

$$\begin{aligned}
\alpha_{i1} &= \sqrt{1 - \tfrac{P_i(1-2v_i)}{2\mu_i(1-v_i)}} \\
\alpha_{i2} &= \sqrt{1 - \tfrac{P_i}{\mu_i}}
\end{aligned} \quad (i = 1, 2). \tag{9}$$

Here A_{ip}, B_{ip}, C_{ip}, D_{ip}, $(i = 1, 2)$, are eight unknown functions of the parameter p. These functions are to be determined by the boundary conditions.

It can be easily demonstrated that the displacement increments expressed by formulas (7) together with Eqs. (8) and (9) satisfy the conditions (3).

Substitution of Eq. (8) into Eq. (7), using the Cauchy relations between the strain and displacement increments and the linear Hooke relations between stress and strain increments leads to the following expressions for the displacement and the stress increments

$$\begin{aligned}
u_i &= \tfrac{2}{\pi}\sum_{p=1}^{\infty}p^2\left(A_{ip}\alpha_{i1}e^{p\alpha_{i1}y} + B_{ip}\alpha_{i2}e^{p\alpha_{i2}y} - C_{ip}\alpha_{i1}e^{-p\alpha_{i1}y} - D_{ip}\alpha_{i2}e^{-p\alpha_{i2}y}\right)\sin px \\
v_i &= -\tfrac{2}{\pi}\sum_{p=1}^{\infty}p^2\left(A_{ip}\alpha_{i2}^2e^{p\alpha_{i1}y} + B_{ip}e^{p\alpha_{i2}y} + C_{ip}\alpha_{i1}^2e^{-p\alpha_{i1}y} + D_{ip}e^{-p\alpha_{i2}y}\right)\cos px
\end{aligned} \quad (i = 1, 2), \tag{10}$$

$$\begin{aligned}\sigma_{iy} &= \tfrac{2}{\pi}\sum_{p=1}^{\infty} p^3\left(A_{ip}\theta_i e^{p\alpha_{i1}y} + B_{ip}\zeta_i e^{p_{i2}y} - C_{ip}\theta_i e^{-p\alpha_{i1}y} - D_{ip}\zeta_i e^{-p\alpha_{i2}y}\right)\cos px \\ \sigma_{ixy} &= \tfrac{2}{\pi}\sum_{p=1}^{\infty} p^3\left[2A_{ip}\alpha_{i1}^2 e^{p\alpha_{i1}y} + B_{ip}(1+\alpha_{i2}^2)e^{p\alpha_{i2}y} + 2C_{ip}\alpha_{i1}^2 e^{-p\alpha_{i1}y} + D_{ip}(1+\alpha_{i2}^2)e^{-p\alpha_{i2}y}\right]\sin px\end{aligned} \quad (i=1,2), \tag{11}$$

where

$$\begin{aligned}\theta_i &= \frac{2\mu_i}{1-2v_i}\left(v_i\alpha_{i1} - (1-v_i)\alpha_{i1}^3\right), \\ \zeta_i &= -2\mu_i\alpha_{i2}.\end{aligned} \tag{12}$$

Substitution of Eqs. (10) and (11) into Eqs. (4) and (6) yields the following set of equations

$$\begin{aligned}&A_{1p}\theta_1 e^{p\alpha_{11}h_1} + B_{1p}\zeta_1 e^{p\alpha_{12}h_1} - C_{1p}\theta_1 e^{-p\alpha_{11}h_1} - D_{1p}\zeta_1 e^{-p\alpha_{12}h_1} = 0, \\ &2A_{1p}\alpha_{11}^2 e^{p\alpha_{11}h_1} + B_{1p}(1+\alpha_{12}^2)e^{p\alpha_{12}h_1} + 2C_{1p}\alpha_{11}^2 e^{-p\alpha_{11}h_1} + D_{1p}(1+\alpha_{12}^2)e^{-p\alpha_{12}h_1} = 0, \\ &A_{2p}\theta_2 e^{-p\alpha_{21}h_2} + B_{2p}\zeta_2 e^{-p\alpha_{22}h_2} - C_{2p}\theta_2 e^{p\alpha_{21}h_2} - D_{2p}\zeta_2 e^{p\alpha_{22}h_2} = 0, \\ &2A_{2p}\alpha_{21}^2 e^{-p\alpha_{21}h_2} + B_{2p}(1+\alpha_{22}^2)e^{-p\alpha_{22}h_2} + 2C_{2p}\alpha_{21}^2 e^{p\alpha_{21}h_2} + D_{2p}(1+\alpha_{22}^2)e^{p\alpha_{22}h_2} = 0, \\ &A_{1p}\theta_1 + B_{1p}\zeta_1 - C_{1p}\theta_1 - D_{1p}\zeta_1 = A_{2p}\theta_2 + B_{2p}\zeta_2 - C_{2p}\theta_2 - D_{2p}\zeta_2, \\ &\frac{\mu_1}{\mu_2}\left(2A_{1p}\alpha_{11}^2 + B_{1p}(1+\alpha_{12}^2) + 2C_{1p}\alpha_{11}^2 + D_{1p}(1+\alpha_{12}^2)\right) = 2A_{2p}\alpha_{21}^2 + B_{2p}(1+\alpha_{22}^2) \\ &\qquad\qquad + 2C_{2p}\alpha_{21}^2 + D_{2p}(1+\alpha_{22}^2), \\ &\frac{2}{\pi}\sum_{p=1}^{\infty} p^2\left(A_{1p}\alpha_{11} - A_{2p}\alpha_{21} + B_{1p}\alpha_{12} - B_{2p}\alpha_{22} - C_{1p}\alpha_{11} + C_{2p}\alpha_{21} - D_{1p}\alpha_{12} + D_{2p}\alpha_{22}\right)\sin px = 0, \\ &\frac{2}{\pi}\sum_{p=1}^{\infty} p^2\left(A_{1p}\alpha_{11}^2 - A_{2p}\alpha_{21}^2 + B_{1p} - B_{2p} + C_{1p}\alpha_{11}^2 - C_{2p}\alpha_{21}^2 + D_{1p} - D_{2p}\right)\cos px = 0, \quad |x| > a.\end{aligned} \tag{13}$$

Further, we introduce the following two unknown functions

$$\begin{aligned}f_u(x) &= \frac{d}{dx}[u_1(x,0) - u_2(x,0)], \\ f_v(x) &= \frac{d}{dx}[v_1(x,0) - v_2(x,0)].\end{aligned} \tag{14}$$

It is clear, that

$$f_u(x) = 0 \quad \text{and} \quad f_v(x) = 0 \ \text{for}\ |x| > a. \tag{15}$$

Substitution of Eq. (10) into Eq. (14), using Eq. (15) and applying Fourier transforms yields

$$\begin{aligned}&p^2(A_{1p}\alpha_{11} - A_{2p}\alpha_{21} + B_{1p}\alpha_{12} - B_{2p}\alpha_{22} - C_{1p}\alpha_{11} + C_{2p}\alpha_{21} - D_{1p}\alpha_{12} + D_{2p}\alpha_{22}) = F_u(p), \\ &-p^2\left(A_{1p}\alpha_{11}^2 - A_{2p}\alpha_{21}^2 + B_{1p} - B_{2p} + C_{1p}\alpha_{11}^2 - C_{2p}\alpha_{21}^2 + D_{1p} - D_{2p}\right) = F_v(p),\end{aligned} \tag{16}$$

where

$$\begin{aligned}F_u(p) &= \frac{1}{p}\int_0^a f_u(\xi)\cos p\xi\, d\xi, \\ F_v(p) &= \frac{1}{p}\int_0^a f_v(\xi)\sin p\xi\, d\xi.\end{aligned} \tag{17}$$

From the first six equations of Eq. (13) and Eq. (16) one can express the eight unknown functions A_{ip}, B_{ip},C_{ip}, D_{ip}, $(i = 1, 2)$, in the form

$$\begin{aligned} A_{1p} &= C_{1p}R_{11}(p) + D_{1p}R_{12}(p), \\ B_{1p} &= C_{1p}R_{21}(p) + D_{1p}R_{22}(p), \\ A_{2p} &= C_{1p}R_{51}(p) + D_{1p}R_{52}(p), \\ B_{2p} &= C_{1p}R_{61}(p) + D_{1p}R_{62}(p), \\ C_{2p} &= C_{1p}R_{31}(p) + D_{1p}R_{32}(p), \\ D_{2p} &= C_{1p}R_{41}(p) + D_{1p}R_{42}(p), \end{aligned} \tag{18}$$

$$C_{1p} = \frac{1}{p^2\Delta(p)}[F_u(p)R_{82}(p) - F_v(p)R_{72}(p)],$$

$$D_{1p} = \frac{1}{p^2\Delta(p)}[F_v(p)R_{71}(p) - F_u(p)R_{81}(p)],$$

where

$$\Delta(p) = R_{71}(p)R_{82}(p) - R_{81}(p)R_{72}(p). \tag{19}$$

The values of R_{ij}, $(i = 1, 2, \ldots, 8; j = 1, 2)$, are functions of material constants, compressive loads and the parameter p, their expressions are presented in the appendix.

Further, substitution of Eq. (11) into conditions (5) gives for $|x| < a$

$$\begin{aligned} &\frac{2}{\pi}\sum_{p=1}^{\infty} p^3\left(A_{1p}\theta_1 + B_{1p}\zeta_1 - C_{1p}\theta_1 - D_{1p}\zeta_1\right)\cos px = 0, \\ &\frac{2}{\pi}\sum_{p=1}^{\infty} p^3[2A_{1p}\alpha_{11}^2 + B_{1p}(1+\alpha_{12}^2) + 2C_{1p}\alpha_{11}^2 + D_{1p}(1+\alpha_{12}^2)]\sin px = 0. \end{aligned} \tag{20}$$

Then substitution of expressions for A_{1p}, B_{1p}, C_{1p}, D_{1p} of Eq. (18) into the first equation of Eq. (20) leads to the following relation

$$\frac{2}{\pi}\sum_{p=1}^{\infty} p[F_u(p)Q_{11}(p) + F_v(p)Q_{12}(p)]\cos px = 0, \quad |x| < a, \tag{21}$$

where the functions Q_{1j} depend on the material constants μ_i, v_i and the compressive loads P_i and they are presented in the appendix.

Applying Fourier transforms to Eq. (21) and expressing $Q_{1j}(p)$ as

$$Q_{1j}(p) = (Q_{1j}(p)Q_{1j}^{\infty}) + Q_{1j}^{\infty}, \tag{22}$$

with

$$Q_{1j}^{\infty} = \lim_{p\to\infty} Q_{1j}(p),$$

Eq. (21) can be written as

$$\beta_1 f_u(x) + \frac{1}{\pi}\int_{-a}^{a}\sum_{p=1}^{\infty}\sin p\xi \cos px\, f_v(\xi)\,\mathrm{d}\xi + \frac{1}{\pi}\int_{-a}^{a}[f_u(\xi)K_{11}(\xi,x) + f_v(\xi)K_{12}(\xi,x)]\,\mathrm{d}\xi = 0, \tag{23}$$

where β_1 and K_{1j} depend on μ_i, v_i, P_i and they are given in the appendix.

To extract the singularity from the second term of (23) we take into account that the function $f_v(\xi)$ is odd and transform this term in the following manner

$$
\begin{aligned}
\frac{1}{\pi}\int_{-a}^{a}\sum_{p=1}^{\infty}(\sin p\xi\cos px-\cos p\xi\sin px)f_v(\xi)\,\mathrm{d}\xi &= \frac{1}{\pi}\int_{-a}^{a}\sum_{p=1}^{\infty}\sin p(\xi-x)f_v(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{\pi}\int_{-a}^{a}\frac{1}{2}\frac{\sin(\xi-x)}{1-\cos(\xi-x)}f_v(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{-a}^{a}\mathrm{ctg}\frac{\xi-x}{2}f_v(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{-a}^{a}f_v(\xi)\left(\mathrm{ctg}\frac{\xi-x}{2}-\frac{2}{\xi-x}\right)\mathrm{d}\xi+\frac{1}{\pi} \\
&\quad\times\int_{-a}^{a}f_v(\xi)\frac{1}{\xi-x}\,\mathrm{d}\xi. \qquad (24)
\end{aligned}
$$

Applying a similar approach to the second of Eq. (20) the system of homogeneous Cauchy-type singular integral equations of the second kind is obtained in the form

$$
\begin{aligned}
&\beta_1 f_u(x)+\frac{1}{\pi}\int_{-a}^{a}\frac{f_v(\xi)}{\xi-x}\,\mathrm{d}\xi+\frac{1}{2\pi}\int_{-a}^{a}f_v(\xi)\left(\mathrm{ctg}\frac{\xi-x}{2}-\frac{2}{\xi-x}\right)\mathrm{d}\xi \\
&\quad+\frac{1}{\pi}\int_{-a}^{a}[f_u(\xi)K_{11}(\xi,x)+f_v(\xi)K_{12}(\xi,x)]\,\mathrm{d}\xi=0, \\
&-\beta_2 f_v(x)+\frac{1}{\pi}\int_{-a}^{a}\frac{f_u(\xi)}{\xi-x}\mathrm{d}\xi+\frac{1}{2\pi}\int_{a}^{a}f_u(\xi)\left(\mathrm{ctg}\frac{\xi-x}{2}-\frac{2}{\xi-x}\right)\mathrm{d}\xi \\
&\quad+\frac{1}{\pi}\int_{-a}^{a}[f_u(\xi)K_{21}(\xi,x)+f_v(\xi)K_{22}(\xi,x)]\mathrm{d}\xi=0, \quad |x|<a,
\end{aligned}
\qquad (25)
$$

where the expressions for the coefficients β_i $(i=1,2)$ depending on the material constants and compressive loads as well as the kernels $K_{ij}(\xi,x)$ $(i,j=1,2)$ are presented in the appendix.

The additional conditions which must be satisfied to ensure, that the displacement be single valued are the following

$$
\begin{aligned}
&\frac{1}{\pi}\int_{-a}^{a}f_u(\xi)\,\mathrm{d}\xi=0, \\
&\frac{1}{\pi}\int_{-a}^{a}f_v(\xi)\,\mathrm{d}\xi=0, \quad |x|<a.
\end{aligned}
\qquad (26)
$$

The buckling load P_{cr}, i.e. the critical value of compressive stress P_1, can be found from the condition of existence of a nontrivial solution of the system (25). To define this value a numerical analysis of the systems (25) and (26) will be applied.

3. Solution of the system of singular integral equations

Substituting $\xi=as$ and $x=at$ one can rewrite the system of integral equation (25) in the form

$$
\begin{aligned}
&\beta_1 f_u^*(t) + \frac{1}{\pi}\int_{-1}^{1}\frac{f_v^*(s)}{s-t}\,\mathrm{d}s + \frac{1}{2\pi}\int_{-1}^{1} f_v^*(s)\left(a\ \mathrm{ctg}\ a\frac{s-t}{2} - \frac{2}{s-t}\right)\mathrm{d}s \\
&\quad + \frac{1}{\pi}\int_{-1}^{1}\left[f_u^*(s)K_{11}^*(s,t) + f_v^*(s)K_{12}^*(s,t)\right]\mathrm{d}s = 0, \\
&-\beta_2 f_v^*(t) + \frac{1}{\pi}\int_{-1}^{1}\frac{f_u^*(s)}{s-t}\,\mathrm{d}s + \frac{1}{2\pi}\int_{-1}^{1} f_u^*(s)\left(a\ \mathrm{ctg}\ a\frac{s-t}{2} - \frac{2}{s-t}\right)\mathrm{d}s \\
&\quad + \frac{1}{\pi}\int_{-1}^{1}\left[f_u^*(s)K_{21}^*(s,t) + f_v^*(s)K_{22}^*(s,t)\right]\mathrm{d}s = 0
\end{aligned}
\tag{27}
$$

and the additional conditions can be presented as

$$
\begin{aligned}
&\frac{1}{\pi}\int_{-1}^{1} f_u^*(s)\,\mathrm{d}s = 0, \\
&\frac{1}{\pi}\int_{-1}^{1} f_v^*(s)\,\mathrm{d}s = 0,
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
&f_u^*(s) = f_u(as), \qquad f_v^*(s) = f_v(as), \\
&K_{ij}^*(s,t) = aK_{ij}(as, at) \quad (i,j = 1,2) \ \text{and} \ |t| < 1.
\end{aligned}
\tag{29}
$$

Using the behavior of unknown functions at the crack tips (Williams, 1959) we will find these functions in the form (Wang and Takao, 1995)

$$
\begin{aligned}
&f_u^*(s) = \frac{1}{\sqrt{1-s^2}}\left[g_1^*(s)\cos\theta - g_2^*(s)\sin\theta\right], \\
&f_v^*(s) = \frac{1}{\sqrt{1-s^2}}\left[g_1^*(s)\sin\theta + g_2^*(s)\cos\theta\right],
\end{aligned}
\tag{30}
$$

where

$$
\theta = -\gamma\ln(1-s^2) \quad \text{and} \quad \gamma = \frac{1}{2\pi}\ln\left(\frac{1+\sqrt{\beta_1\beta_2}}{1-\sqrt{\beta_1\beta_2}}\right). \tag{31}
$$

Substituting of Eq. (30) into Eqs. (27) and (28), consider the obtained relations at the points $t_k = \cos(\pi(k/n))$, $(k = 1, 2, \ldots, n-1)$, which are the zeros of the Chebyshev polynomial $U_{n-1}(t)$ (Erdogan and Gupta, 1972). Employing the Gauss–Chebyshev quadrature rule,

$$
\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(s)}{\sqrt{1-s^2}}\frac{\mathrm{d}s}{s-t} \approx \sum_{i=1}^{n}\frac{1}{n}\psi(s_i)\frac{1}{s_i - t_k}
$$

with

$$
s_i = \cos\left(\pi\frac{2i-1}{2n}\right) \quad (i = 1, 2, \ldots, n) \tag{32}
$$

and using the following approximation

$$
\begin{aligned}
&f_u^*(t_k) \approx \frac{1}{2}\left[f_u^*(s_k) + f_u^*(s_{k+1})\right], \\
&f_v^*(t_k) \approx \frac{1}{2}\left[f_v^*(s_k) + f_v^*(s_{k+1})\right],
\end{aligned}
\tag{33}
$$

Eqs. (27) and (28) can be reduced to the following system of $2n$ homogeneous algebraic equations with $2n$ unknown values $g_1^*(si)$ and $g_2^*(si)$ $(i = 1, 2, \dots, n)$:

$$\begin{aligned}
&\frac{\beta_1}{2}\left[g_1^*(s_k)\frac{\cos\theta_k}{\sqrt{1-s_k^2}} + g_1^*(s_{k+1})\frac{\cos\theta_{k+1}}{\sqrt{1-s_{k+1}^2}} - g_2^*(s_k)\frac{\sin\theta_k}{\sqrt{1-s_k^2}} - g_2^*(s_{k+1})\frac{\sin\theta_{k+1}}{\sqrt{1-s_{k+1}^2}}\right]\\
&+\frac{1}{n}\sum_{i=1}^{n}\left\{g_1^*(s_i)\left[\sin\theta_i\left[\frac{1}{s_i-t_k}+\frac{1}{2}\left(a\ \mathrm{ctg}\ a\frac{s_i-t_k}{2}-\frac{2}{s_i-t_k}\right)+K_{12}^*(s_i,t_k)\right]+K_{11}^*(s_i,t_k)\cos\theta_i\right]\right.\\
&\left.+g_2^*(s_i)\left[\cos\theta_i\left[\frac{1}{s_i-t_k}+\frac{1}{2}\left(a\ \mathrm{ctg}\ a\frac{s_i-t_k}{2}-\frac{2}{s_i-t_k}\right)+K_{12}^*(s_i,t_k)\right]-K_{11}^*(s_i,t_k)\sin\theta_i\right]\right\}\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&-\frac{\beta_2}{2}\left[g_1^*(s_k)\frac{\sin\theta_k}{\sqrt{1-s_k^2}} + g_1^*(s_{k+1})\frac{\sin\theta_{k+1}}{\sqrt{1-s_{k+1}^2}} - g_2^*(s_k)\frac{\cos\theta_k}{\sqrt{1-s_k^2}} - g_2^*(s_{k+1})\frac{\cos\theta_{k+1}}{\sqrt{1-s_{k+1}^2}}\right]\\
&+\frac{1}{n}\sum_{i=1}^{n}\left\{g_1^*(s_i)\left[\cos\theta_i\left[\frac{1}{s_i-t_k}+\frac{1}{2}\left(a\ \mathrm{ctg}\ a\frac{s_i-t_k}{2}-\frac{2}{s_i-t_k}\right)+K_{21}^*(s_i,t_k)\right]+K_{22}^*(s_i,t_k)\sin\theta_i\right]\right.\\
&\left.+g_2^*(s_i)\left[-\sin\theta_i\left[\frac{1}{s_i-t_k}+\frac{1}{2}\left(a\ \mathrm{ctg}\ a\frac{s_i-t_k}{2}-\frac{2}{s_i-t_k}\right)+K_{21}^*(s_i,t_k)\right]+K_{22}^*(s_i,t_k)\cos\theta_i\right]\right\}\\
&=0,\frac{1}{n}\sum_{i=1}^{n}\left[g_1^*(s_i)\cos\theta_i-g_2^*(s_i)\sin\theta_i\right]=0,\frac{1}{n}\sum_{i=1}^{n}\left[g_1^*(s_i)\sin\theta_i+g_2^*(s_i)\cos\theta_i\right]=0,
\end{aligned} \tag{34}$$

where

$$\theta_i = -\gamma\ln(1-s_i^2).$$

The buckling load P_{cr} can be found from the condition that the determinant of the obtained system is equal to zero.

4. Numerical analysis

The influence of the stiffness ratio and the geometrical parameter hi/a upon the critical load have been analyzed. The corresponding numerical results are presented in Figs. 2–7. The Table 1 exhibits the behavior P_{cr} with respect to the parameter of discretization n for $\mu_2/\mu_1 = 10$, $\nu_1 = 0.3$, $\nu_2 = 0.2$ and $h_1/a = 0.5$, $h_2/a = 4$, $L/a = 2\pi$. It can be clearly seen that the results of the calculations are in quite good agreement for all values of n and the difference of P_{cr} obtained for $n = 30$ with respect to the associated result for $n = 50$

Table 1
Effect of the value of n defining the rate of SIE discretization upon the buckling load P_{cr} for $\mu_2/\mu_1 = 10$, $\nu_1 = 0.3$, $\nu_2 = 0.2$ and $h_1/a = 0.5$, $h_2/a = 4$, $L/a = 2\pi$

n	20	25	30	35	40	45	50
P_{cr}/μ_1	0.26957	0.26839	0.26775	0.26735	0.26710	0.26693	0.26680

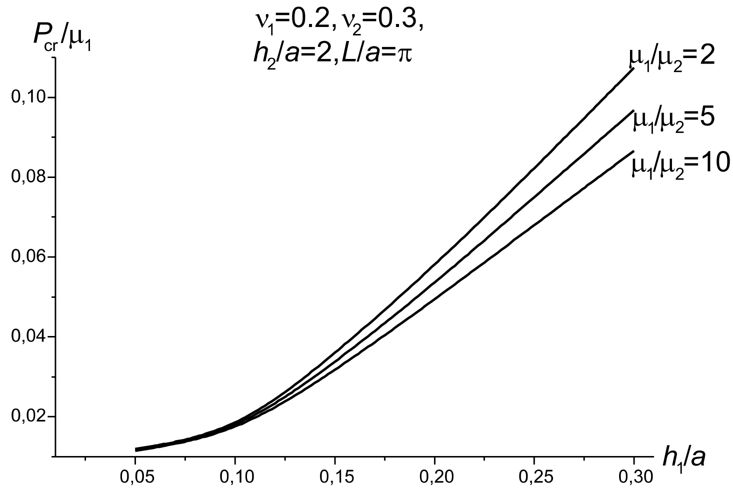

Fig. 2. Variation of the buckling load with the thickness of the layer, for $\mu_1 > \mu_2$, $h_1 < h_2$ and $\nu_1 = 0.2$, $\nu_2 = 0.3$, $h_2/a = 2$, $L/a = \pi$.

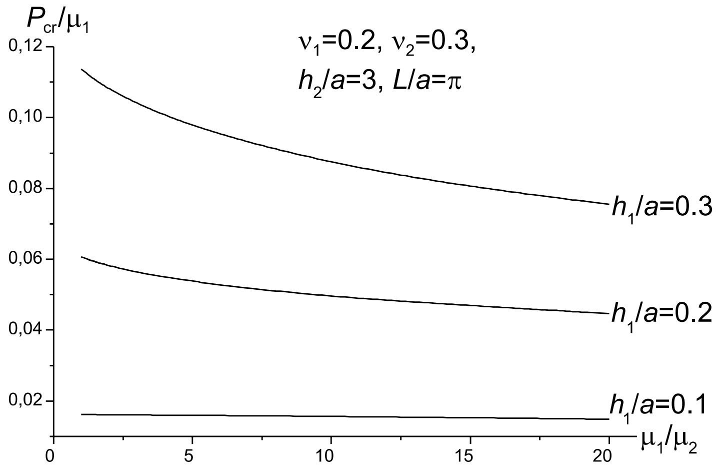

Fig. 3. Variation of the buckling load with the shear modulus ratio, for $\mu_1 > \mu_2$, $h_1 < h_2$ and $\nu_1 = 0.2$, $\nu_2 = 0.3$, $h_2/a = 3$, $L/a = \pi$.

is less than 0.01%. Taking into account that for the other loading the dependence on n is similar to the analyzed case the results in Figs. 2–7 are presented for $n = 30$.

Particularly Fig. 2 describes the critical loads P_{cr}/μ_1 as a function of parameter h_1/a for various shear modulus ratios μ_1/μ_2 ($\mu_1 > \mu_2$) and $\nu_1 = 0.2$, $\nu_2 = 0.3$, $h_2/a = 2$, $L/a = \pi$. It can be seen from these results that decreasing of μ_2 leads for the fixed values of the other parameters to the increasing of P_{cr}/μ_1 and increasing of h_1/a leads to the increasing of P_{cr}/μ_1 as well.

In Figs. 3–6 the critical loads are shown for various geometrical parameters as a function of stiffness ratio. It follows from these figures that the critical load slightly depends on the coefficient μ_1/μ_2 for small values h_1/a (Figs. 3 and 5) and h_2/a (Figs. 4 and 6). On the other hand this load becomes sensitive to the variation of the stiffness coefficient for larger values of the mentioned parameters. Moreover increasing of

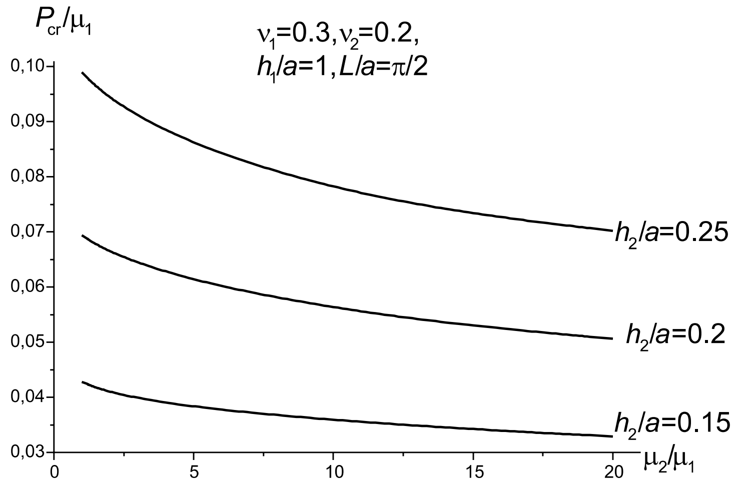

Fig. 4. Variation of the buckling load with the shear modulus ratio, for $\mu_1 < \mu_2$, $h_1 > h_2$ and $\nu_1 = 0.3$, $\nu_2 = 0.2$, $h_1/a = 1$, $L/a = \pi/2$.

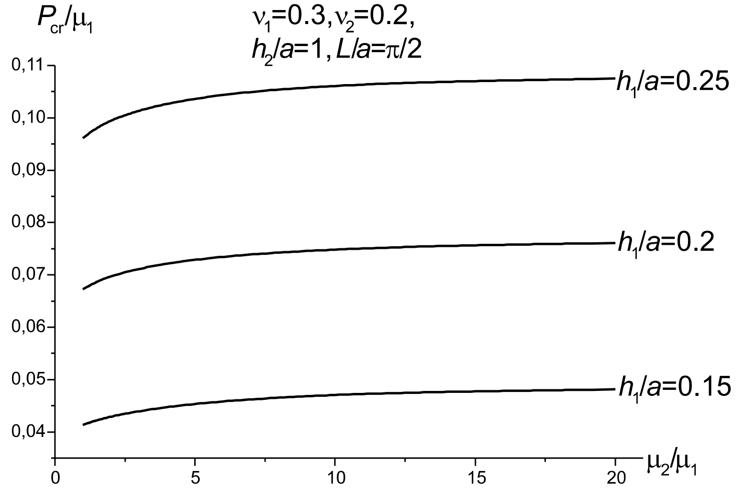

Fig. 5. Variation of the buckling load with the shear modulus ratio, for $\mu_1 < \mu_2$, $h_1 < h_2$ and $\nu_1 = 0.3$, $\nu_2 = 0.2$, $h_1/a = 1$, $L/a = \pi/2$.

h_1/a (Figs. 3 and 5) or h_2/a (Figs. 4 and 6) lead to the essential increasing of P_{cr}/μ_1 that is quite reasonable from the physical point of view.

The values of the critical load has been found for various values of geometrical and mechanical parameters of the rod. Particularly, for the large values of h_2/h_1 and L/a the obtained results almost coincide with the correspondent results of the paper of Wang and Takao (1995). This statement can be confirmed by Fig. 7, where the behavior of P_{cr}/μ_1 with respect to h_2/a is shown for $\nu_1 = 0.3$, $\nu_2 = 0.2$, $\mu_2/\mu_1 = 10$, $h_1/a = 0.5$, $L/a = 5\pi$. It should be mentioned that the correspondent result of Wang and Takao (1995) for the same ν_1, ν_2, μ_2/μ_1, h_1/a and $L/a \to \infty$, $h_2/a \to \infty$ is $P_{cr}/\mu_1 \approx 0.268$. A comparison shows that the values of P_{cr}/μ_1 in Fig. 7 tends to this quality and as early as for $h_2/a = 1.5$ the difference is less than 0.1%.

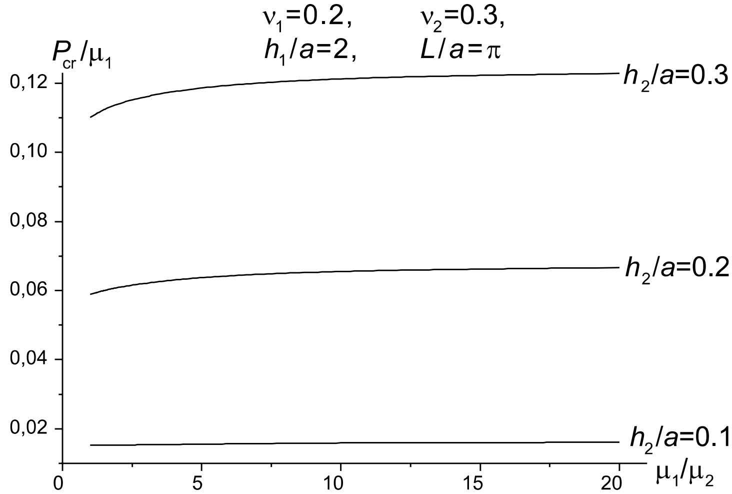

Fig. 6. Variation of the buckling load with the shear modulus ratio, for $\mu_1 > \mu_2$, $h_1 > h_2$ and $\nu_1 = 0.2$, $\nu_2 = 0.3$, $h_1/a = 1$, $L/a = \pi$.

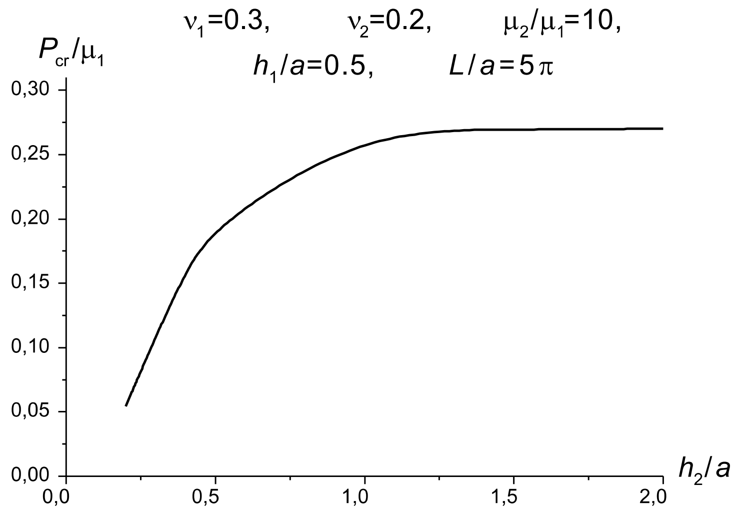

Fig. 7. Variation of the buckling load with respect to the geometrical parameter h_2/a, for $\nu_1 = 0.3$, $\nu_2 = 0.2$, $\mu_2/\mu_1 = 10$, $h_1/a = 0.5$ and $L/a = 5\pi$.

5. Conclusion

In the present paper the plane problem of the stability loss of a finite sized layered system subjected to a uniform compressive load and containing an interface crack is considered. By using finite Fourier transform the problem is reduced to some homogeneous system of singular integral equations with respect to the increment of the displacement jumps across the material interface in the region of the crack. Further, by means of a collocation procedure and Gauss–Chebyshev quadrature rule, the mentioned system is reduced to a system of homogeneous algebraic equations. The critical load has been found from the condition that the determinant of the obtained system is equal to zero.

The numerical results for the critical load demonstrate the influence of the material properties and the geometrical parameters of both layers upon the local delamination buckling behavior of the layered system. By means of the presented algorithm the critical values can be found for any relations between the geometrical and mechanical parameters of a bimaterial rod with an interface crack.

Appendix A

$$R_{11}(p) = \frac{a_{11}}{\Delta_1} e^{-2p\alpha_{11}h_1}, \qquad R_{12}(p) = \frac{a_{31}}{\Delta_1} e^{-p(\alpha_{11}+\alpha_{12})h_1},$$

$$R_{21}(p) = \frac{a_{21}}{\Delta_1} e^{-p(\alpha_{11}+\alpha_{12})h_1}, \qquad R_{22}(p) = \frac{a_{41}}{\Delta_1} e^{-2p\alpha_{12}h_1},$$

$$R_{3i}(p) = \frac{1}{\Delta_3(p)}[V_{4i}(p)V_{52}(p) - V_{32}(p)V_{6i}(p)]e^{-p\alpha_{21}h_2},$$

$$R_{4i}(p) = \frac{1}{\Delta_3(p)}[-V_{4i}(p)V_{51}(p) + V_{31}(p)V_{6i}(p)]e^{-p\alpha_{22}h_2},$$

$$R_{5i}(p) = V_{1i}(p) + R_{3i}(p)\frac{a_{12}}{\Delta_2} + R_{4i}(p)\frac{a_{32}}{\Delta_2},$$

$$R_{6i}(p) = V_{2i}(p) + R_{3i}(p)\frac{a_{22}}{\Delta_2} + R_{4i}(p)\frac{a_{42}}{\Delta_2},$$

$$R_{7i}(p) = R_{1i}(p)\alpha_{11} + R_{2i}(p)\alpha_{12} + R_{3i}(p)\alpha_{21} + R_{4i}(p)\alpha_{22} - R_{5i}(p)\alpha_{21} - R_{6i}(p)\alpha_{22} - \alpha_{1i},$$

$$R_{81}(p) = R_{11}(p)\alpha_{11}^2 + R_{21}(p) - R_{31}(p)\alpha_{21}^2 - R_{42}(p) - R_{51}(p)\alpha_{21}^2 - R_{61}(p) + \alpha_{11}^2,$$

$$R_{82}(p) = R_{12}(p)\alpha_{11}^2 + R_{22}(p) - R_{32}(p)\alpha_{21}^2 - R_{42}(p) - R_{52}(p)\alpha_{21}^2 - R_{62}(p) + 1,$$

$$\Delta_i = \theta_i(1+\alpha_{i2}^2) - 2\zeta_i\alpha_{i1}^2,$$

$$a_{1i} = \theta_i(1+\alpha_{i2}^2) + 2\zeta_i\alpha_{i1}^2, \qquad a_{3i} = 2\zeta_i(1+\alpha_{i2}^2),$$

$$a_{2i} = -4\theta_i\alpha_{i1}^2, \qquad a_{4i} = -a_{1i},$$

$$b_{11} = \theta_1(1+\alpha_{22}^2) - 2\zeta_2\alpha_{11}^2\frac{\mu_1}{\mu_2}, \qquad b_{12} = 2\left(\theta_2\alpha_{11}^2\frac{\mu_1}{\mu_2} - \theta_1\alpha_{21}^2\right),$$

$$b_{21} = \zeta_1(1+\alpha_{22}^2) - \zeta_2(1+\alpha_{12}^2)\frac{\mu_1}{\mu_2}, \qquad b_{22} = \theta_{2c}(1+\alpha_{12}^2)\frac{\mu_1}{\mu_2} - 2\zeta_1\alpha_{21}^2,$$

$$b_{31} = -\theta_1(1+\alpha_{22}^2) - 2\zeta_2\alpha_{11}^2\frac{\mu_1}{\mu_2}, \qquad b_{32} = 2\left(\theta_2\alpha_{11}^2\frac{\mu_1}{\mu_2} + \theta_{1c}\alpha_{21}^2\right),$$

$$b_{41} = -\zeta_1(1+\alpha_{22}^2) - \zeta_2(1+\alpha_{12}^2)\frac{\mu_1}{\mu_2}, \qquad b_{42} = \theta_2(1+\alpha_{12}^2)\frac{\mu_1}{\mu_2} + 2\zeta_1\alpha_{21}^2,$$

$$V_{i1}(p) = \frac{1}{\Delta_2}[b_{1i}R_{11}(p) + b_{2i}R_{21}(p) + b_{3i}],$$

$$V_{i2}(p) = \frac{1}{\Delta_2}[b_{1i}R_{12}(p) + b_{2i}R_{22}(p) + b_{4i}],$$

$$V_{31}(p) = \frac{a_{12}}{\Delta_2}\theta_2 e^{-2p\alpha_{21}h_2} + \frac{a_{22}}{\Delta_2}\zeta_2 e^{-p(\alpha_{21}+\alpha_{22})h_2} - \theta_2,$$

$$V_{32}(p) = \frac{a_{32}}{\Delta_2}\theta_2 e^{-p(\alpha_{21}+\alpha_{22})h_2} + \frac{a_{42}}{\Delta_2}\zeta_2 e^{-2p\alpha_{22}h_2} - \zeta_2,$$

$$V_{4i}(p) = -V_{1i}(p)\theta_2 e^{-p\alpha_{21}h_2} - V_{2i}(p)\zeta_2 e^{-p\alpha_{22}h_2},$$

$$V_{51}(p) = 2\frac{a_{12}}{\Delta_2}\alpha_{21}^2 e^{-2p\alpha_{21}h_2} + \frac{a_{22}}{\Delta_2}(1+\alpha_{22}^2)e^{-p(\alpha_{21}+\alpha_{22})h_2} + 2\alpha_{21}^2,$$

$$V_{52}(p) = 2\frac{a_{32}}{\Delta_2}\alpha_{21}^2 e^{-p(\alpha_{21}+\alpha_{22})h_2} + \frac{a_{42}}{\Delta_2}(1+\alpha_{22}^2)e^{-2p\alpha_{22}h_2} + 1 + 2\alpha_{22}^2,$$

$$V_{6i}(p) = -2V_{1i}(p)\alpha_{21}^2 e^{-p\alpha_{21}h_2} - V_{2i}(p)(1+\alpha_{22}^2)e^{-p\alpha_{22}h_2},$$

$$\Delta_3(p) = V_{31}(p)V_{52}(p) - V_{51}(p)V_{32}(p),$$

$$\beta_1 = \frac{Q_{11}^\infty}{Q_{12}^\infty}, \qquad \beta_2 = \frac{Q_{22}^\infty}{Q_{21}^\infty},$$

$$K_{11}(\xi, x) = \frac{1}{Q_{12}^\infty}\sum_{p=1}^{\infty}[Q_{11}(p) - Q_{11}^\infty]\cos p\xi \cos px,$$

$$K_{12}(\xi, x) = \frac{1}{Q_{12}^\infty}\sum_{p=1}^{\infty}[Q_{12}(p) - Q_{12}^\infty]\sin p\xi \cos px,$$

$$K_{21}(\xi, x) = -\frac{1}{Q_{21}^\infty}\sum_{p=1}^{\infty}[Q_{21}(p) - Q_{21}^\infty]\cos p\xi \sin px,$$

$$K_{22}(\xi, x) = -\frac{1}{Q_{22}^\infty}\int_{p=1}^{\infty}[Q_{22}(p) - Q_{22}^\infty]\sin p\xi \sin px,$$

where

$$Q_{i1}(p) = \frac{1}{\Delta(p)}[N_{i2}(p)R_{81}(p) - N_{i1}(p)R_{82}(p)],$$

$$Q_{i2}(p) = \frac{1}{\Delta(p)}[N_{i1}(p)R_{72}(p) - N_{i2}(p)R_{71}(p)], \quad (i = 1, 2),$$

$$N_{11}(p) = \theta_1 R_{11}(p) + \zeta_1 R_{21}(p) - \theta_1,$$

$$N_{12}(p) = \theta_1 R_{12}(p) + \zeta_1 R_{22}(p) - \zeta_1,$$

$$N_{21}(p) = 2\alpha_{11}^2 R_{11}(p) + (1 + \alpha_{12}^2) R_{21}(p) + 2\alpha_{11}^2,$$

$$N_{22}(p) = 2\alpha_{11}^2 R_{12}(p) + (1 + \alpha_{12}^2) R_{22}(p) + (1 + \alpha_{12}^2).$$

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